Computer Assisted Method for Proving Existence of Periodic Orbits

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Abstract

We introduce a method based on the Conley index theory for proving the existence of a periodic trajectory in a smooth dynamical system in $\mathbb{R}^n$ where an attracting periodic orbit is numerically observed. We apply this method to prove the existence of a periodic orbit in the Rössler equations, as announced in [9].

1 Introduction

The aim of this paper is to give a computer assisted proof of the following theorem:

Theorem 1 *The Rössler system [10]*

$$\begin{aligned} \dot{x} &= -(y+z), \\ \dot{y} &= x + by, \\ \dot{z} &= b + z(x-a) \end{aligned} \tag{1}$$

for $a = 2.2$ *and* $b = 0.2$ *admits a periodic orbit.*

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The existence of such a periodic orbit was conjectured by Hale and Koçak in [2], where also numerical evidence of the existence of the periodic orbit in this system was given.

Periodic solutions to differential equations in $\mathbb{R}^n$ are among the basic objects of interest in the theory of dynamical systems. A well known approach in proving the existence of periodic orbits is to construct a Poincaré section and to use suitable topological theorems to find a fixed point of the Poincaré map. However, as observed in [2], it is usually difficult to locate a Poincaré section and even worse to compute the Poincaré map, which makes these methods difficult to apply.

In this paper we apply another method for proving the existence of periodic orbits. The method is based on the Conley index theory and does not require the analysis of the Poincaré map. It uses recently developed and implemented algorithms for computation of homology of representable sets and maps, together with a method for rigorous integration of differential equations. The proposed method is quite general. It may be applied to an arbitrary autonomous differential equation in $\mathbb{R}^n$ which exhibits in numerical simulations an attracting periodic trajectory.

The approach is constructive in the sense that the periodic orbit is proved to be in an effectively constructed neighbourhood of the numerically observed trajectory. This neighbourhood is obtained as a set built of hypercubes and may be a subject of further analysis or visualization.

The method introduced here is based on several theorems from [6] and [7]. They are proved there in the context of a semiflow on an arbitrary metric space. In order to avoid complicated definitions not really needed here, we reformulate these theorems in our context. For proofs, details and generalizations the interested reader is referred to [6] and [7].

2 Preliminaries

Consider the differential equation

$$\dot{x} \;=\; f(x), \tag{2}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field of class C^2.

The *continuous dynamical system* (also called a *flow*) generated by the differential equation (2) is a function $\varphi : \mathbb{R}^n \times \mathbb{R} \ni (x,t) \mapsto \varphi(x,t) \in \mathbb{R}^n$

such that for every $x \in \mathbb{R}^n$ the function $\varphi(x, \cdot) : \mathbb{R} \to \mathbb{R}^n$ is a solution to (2) with $\varphi(x, 0) = x$. For a fixed $t > 0$, a *discretization*, called also a *time-t map*, of the flow φ is a restriction of φ to $\mathbb{R}^n \times t\mathbb{Z}$. This restriction, denoted by $\varphi_t : \mathbb{R}^n \times t\mathbb{Z} \to \mathbb{R}^n$, is the *discrete dynamical system* generated by φ_t, i.e., $\varphi_t(x, kt) = \varphi_t^k(x)$ for every $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$ (in particular, $\varphi_t(\cdot, 0)$ is the identity).

For a set $N \subset \mathbb{R}^n$ its *invariant part* with respect to the flow φ is defined as $\operatorname{inv}(N, \varphi) = \{x \in N \mid \varphi(x, \mathbb{R}) \subset N\}$ or, in an equivalent way, $\operatorname{inv}(N, \varphi) = \bigcap_{t \in \mathbb{R}} \varphi(N, t)$. The set N is called an *isolating neighbourhood* if it is compact and $\operatorname{inv}(N, \varphi) \subset \operatorname{int} N$. In such a case, $S = \operatorname{inv}(N, \varphi)$ is called an *isolated invariant set* and N is said to *isolate* S. The same definitions may be formulated for a discretization φ_t of the flow φ starting with the definition of $\operatorname{inv}(N, \varphi_t)$ obtained from the definition of $\operatorname{inv}(N, \varphi)$ by replacing $\mathbb{R}$ with $t\mathbb{Z}$.

A compact subset Ξ of an $(n-1)$–dimensional hyperplane P is called a *local section* for φ if the vector field f is transverse to P on Ξ. Such a set Ξ is called a *Poincaré section* for φ in an isolating neighbourhood N if $\Xi \cap N$ is closed and for every $x \in N$ there exists $t > 0$ such that $\varphi(x, t) \in \Xi$.

The definitions of the *Conley index* of an isolating neighbourhood N and an isolated invariant set S both in the discrete and in the continuous case are based on the notion of an index pair. The reader is referred to [6] or [7] for details. The following proposition follows easily from the definition of the Conley index:

Proposition 2 *Let N be an isolating neighbourhood of a discretization φ_t of the flow φ. If $\varphi_t(N) \subset N$, the cohomology of N is the cohomology of S^1 and the map φ_t with its domain and range restricted to N induces an isomorphism in cohomology then N has the cohomological Conley index of an attracting periodic orbit.*

To prove the existence of a periodic orbit we verify the assumptions of the following theorem, which is a special case of Corollary 1.4 in [6]:

Theorem 3 *Assume N is an isolating neighbourhood for the flow φ which admits a Poincaré section Ξ. If N has the cohomological Conley index of a hyperbolic periodic orbit then* $\operatorname{inv}(N, \varphi)$ *contains a periodic orbit.*

Since we perform computations for a discretization φ_t of the flow φ with an arbitrarily chosen $t > 0$, we need the following two theorems, proved in a more general setting in [7] as Theorem 1 and Corollary:

Theorem 4 *For a flow φ in $\mathbb{R}^n$ the following three conditions are equivalent:*
(1) S is an isolated invariant set with respect to φ,
(2) S is an isolated invariant set with respect to φ_t for all $t > 0$,
(3) S is an isolated invariant set with respect to φ_t for some $t > 0$.

Theorem 5 *The cohomological Conley index of an isolated invariant set of a flow φ coincides with the corresponding index with respect to the discrete dynamical system φ_t for any $t > 0$.*

Let us now introduce the class of representable sets and maps we work with. We recall that a Cartesian product $\Delta_1 \times \cdots \times \Delta_n$ of n compact intervals in $\mathbb{R}^n$ is called an n-dimensional hypercube. Such a hypercube is of size $d \geq 0$ if each interval Δ_i has length d. We fix a grid size $d > 0$ and we denote by $\mathcal{H}$ the set of all closed n-dimensional hypercubes of size d in $\mathbb{R}^n$ with vertices in $(d\mathbb{Z})^n$. For $\mathcal{A} \subset \mathcal{H}$ let $|\mathcal{A}|$ denote the union $\bigcup \mathcal{A} = \bigcup_{a \in \mathcal{A}} a$. Any subset A of $\mathbb{R}^n$ such that $A = |\mathcal{A}|$ for some $\mathcal{A} \subset \mathcal{H}$ is called a *cubical set*.

For subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{H}$, a map $\mathcal{F} : \mathcal{A} \to 2^{\mathcal{B}}$, which maps each hypercube in $\mathcal{A}$ to a nonempty set of hypercubes in $\mathcal{B}$, is called a *multivalued cubical map*. Such a map $\mathcal{F}$ is called *finite* if $\mathcal{A}$ is finite and for each $a \in \mathcal{A}$ the set $\mathcal{F}(a)$ is finite. A finite map $\mathcal{F}$ is called *almost perfect* if for each $a \in \mathcal{A}$ the set $|\mathcal{F}(a)|$ is convex. The multivalued map $|\mathcal{F}| : |\mathcal{A}| \ni x \mapsto \bigcup_{\{a \in \mathcal{A} | x \in a\}} |\mathcal{F}(a)| \in 2^{|\mathcal{B}|}$ is then almost perfect in the sense of the definition used in [1]. A finite multivalued cubical map $\mathcal{F} : \mathcal{A} \to 2^{\mathcal{B}}$ is called a *cubical enclosure* of a continuous map $f : |\mathcal{A}| \to |\mathcal{B}|$ if $f(a) \subset \operatorname{int} |\mathcal{F}(a)|$ for each $a \in \mathcal{A}$. If $\mathcal{F}$ is an almost perfect cubical enclosure of f then the chain map of $|\mathcal{F}|$ is defined in [1] as a chain map ψ between the chain complexes of $|\mathcal{A}|$ and $|\mathcal{B}|$ such that the homology of ψ coincides with the homology of f. An algorithm for the construction of a suitable map ψ is given in [1] and was recently implemented by Mazur and Szybowski [5].

To compute a cubical enclosure $\mathcal{F}$ of φ_t on a finite set $\mathcal{A} \in \mathcal{H}$, we use one of Lohner's methods for computation of guaranteed enclosures for the solutions of ordinary initial value problems [4], recently analyzed and implemented by Mrozek and Zgliczyński [8]. This is a method for computing enclosures for φ_τ for $\tau > 0$, which works with sets of the form $x + C\mathbf{r}_0 + \mathbf{s}$, where x is a vector, C is a matrix, $\mathbf{r}_0$ is an interval vector, and $\mathbf{s}$ is a small set from a certain class of representable sets. For more details the reader is referred to [8], where such sets are called doubletons, or to [4], where the corresponding method is called an inner enclosure. Let us only mention that we take an arbitrary $\tau > 0$ such that $t = k\tau$ for a fixed $k \in \mathbb{Z}$, we compute the enclosure for $\varphi_t(a)$ on each $a \in \mathcal{A}$ in k steps as the enclosure for $\varphi_\tau^k(a)$, and we replace the resulting set with a minimal convex cubical set containing it. The above method will be briefly referred to as Lohner's method.

In the sequel we use the same symbol $\mathcal{F}$ to denote cubical enclosures of various restrictions of φ_t if the domain and range of each restriction is clear from the context.

Finite subsets of $\mathcal{H}$, finite multivalued cubical maps, finitely generated chain complexes, their homology modules, as well as maps between these objects may be easily implemented using aggregate data structures available in programming languages and thus we use them in arguments of algorithms and as returned values.

Let us now recall some basic terminology from the homotopy theory. Let X and Y be topological spaces. We say that two continuous maps $f, g : X \to Y$ are *homotopic*, and we write $f \simeq g$, if there exists a continuous map $h : X \times [0,1] \to Y$ such that $h(\cdot, 0) = f$ and $h(\cdot, 1) = g$. The relation "$\simeq$" is obviously an equivalence. The map h is called a *homotopy* between f and g. We say that a topological space X *has the homotopy type* of a topological space Y, and we write $X \simeq Y$ (as this relation is also an equivalence), if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$, where id_X and id_Y denote the appropriate identity maps. We say that $A \subset X$ is a *retract* of X if there exists a continuous map $r : X \to A$ such that $r(a) = a$ for each $a \in A$. The map r is called a *retraction* of X onto A. We say that A is a *deformation retract* of X if A is a retract of X and $\mathrm{id}_X \simeq i_A \circ r$, i.e., the identity on X is homotopic with a certain retraction $r : X \to A$ composed with the inclusion i_A of A into X. The homotopy between id_X and $i_A \circ r$ is called a *deformation retraction* of X onto A. If there exists a deformation retraction h such that $h(a, t) = a$ for each $a \in A$ and $t \in [0, 1]$ then the set A is called a *strong deformation retract* of X and h is called a *strong deformation retraction*.

3 Construction of an Isolating Neighbourhood

In this section an algorithm for construction of an isolating neighbourhood N for which $\varphi_t(N) \subset \operatorname{int} N$ is described in detail.

Lemma 6 *Let N be an isolating neighbourhood with respect to a discretization φ_t of the flow φ. Then N is an isolating neighbourhood with respect to the flow φ and* $\operatorname{inv}(N, \varphi_t) = \operatorname{inv}(N, \varphi)$.

Proof. Let $\operatorname{inv}(N, \varphi_t)$ be denoted by S. Since S is an isolated invariant set with respect to φ_t, it follows from Theorem 4 that S is an isolated invariant set with respect to the flow φ. Since $\operatorname{inv}(N, \varphi)$ is the maximal invariant set with respect to φ contained in N, it is obvious that $S \subset \operatorname{inv}(N, \varphi)$. On the other hand, we have

$$\operatorname{inv}(N, \varphi) = \bigcap_{s \in \mathbb{R}} \varphi_s(N) \subset \bigcap_{s \in t\mathbf{Z}} \varphi_s(N) = \operatorname{inv}(N, \varphi_t).$$

This means that $\operatorname{inv}(N, \varphi) \subset S$ and we obtain $\operatorname{inv}(N, \varphi_t) = \operatorname{inv}(N, \varphi)$. Since $\operatorname{inv}(N, \varphi_t) \subset \operatorname{int} N$, it follows that also $\operatorname{inv}(N, \varphi) \subset \operatorname{int} N$. Hence N is an isolating neighbourhood with respect to the flow φ. □

Algorithm 7
function *neighbourhood* ($\mathcal{N}$: finite subset of $\mathcal{H}$,
    $\mathcal{F} : \mathcal{A} \to 2^{\mathcal{H}}$: finite multivalued cubical map): finite subset of $\mathcal{H}$;
begin
    if $\mathcal{F}(\mathcal{N}) \subset \mathcal{N}$ **then**
        return $\mathcal{N}$
    else if $\mathcal{F}(\mathcal{N}) \subset \mathcal{A}$ **then**
        return *neighbourhood* ($\mathcal{N} \cup \mathcal{F}(\mathcal{N})$, $\mathcal{F}$)
    else
        return $\emptyset$
end;

Proposition 8 *Let $\mathcal{N}_1 \subset \mathcal{A} \subset \mathcal{H}$, and let $\mathcal{F}$ be a cubical enclosure of φ_t on $|\mathcal{A}|$. Assume Algorithm 7 called with $\mathcal{N}_1$ and $\mathcal{F}$ stops and returns a nonempty set $\mathcal{N} \subset \mathcal{H}$. Then $N := |\mathcal{N}|$ is an isolating neighbourhood with respect to φ_t and φ such that $\varphi_t(N) \subset \operatorname{int} N$ and* $\operatorname{inv}(N, \varphi_t) = \operatorname{inv}(N, \varphi)$.

Proof. Let $\mathcal{N}_k$ denote the set of hypercubes passed to the procedure *neighbourhood* as the first argument in its k-th recursive call. Since we assume that the algorithm does not loop infinitely and does not return an empty set, there exists $K > 0$ such that $\mathcal{F}(\mathcal{N}_K) \subset \mathcal{N}_K \subset \mathcal{A}$, and the algorithm returns

$\mathcal{N} = \mathcal{N}_K$. Since $\mathcal{F}$ is a cubical enclosure of φ_t, we have $\varphi_t(N) \subset \operatorname{int} N$. Since obviously $\operatorname{inv}(N, \varphi_t) \subset \varphi_t(N)$, we have $\operatorname{inv}(N, \varphi_t) \subset \operatorname{int} N$. Moreover, N is compact as a finite union of hypercubes. This implies that N is an isolating neighbourhood with respect to the discretization φ_t of the flow φ. The remaining properties of N follow immediately from Lemma 6. □

4 Reduction of a Cubical Set

The isolating neighbourhood constructed by Algorithm 7 may turn out to be very large. Therefore it is worth to try to reduce it with no change to its (co)homological properties. An algorithm suitable for this purpose is described in this section.

Lemma 9 *Let $a \in \mathcal{H}$ and $\mathcal{C} \subset \mathcal{H}$ be such that $a \notin \mathcal{C}$ and $a \cap b \neq \emptyset$ for each $b \in \mathcal{C}$. Then $a \cap |\mathcal{C}|$ is a strong deformation retract of $|\mathcal{C}|$.*

Proof. Since a is convex, for each $x \in |\mathcal{C}|$ there exists a unique $y(x) \in a$ such that $\operatorname{dist}(x, y(x)) = \operatorname{dist}(x, a)$. It is a simple geometrical matter to see that the map $x \mapsto y(x)$ is continuous and if $x \in b \in \mathcal{C}$ then $y(x) \in b$. The function $|\mathcal{C}| \times [0,1] \ni (x,t) \mapsto (1-t)x + ty(x) \in |\mathcal{C}|$ is then a strong deformation retraction of $|\mathcal{C}|$ onto $a \cap |\mathcal{C}|$, which completes the proof. □

Algorithm 10
function *reduce* ($\mathcal{A}$, $\mathcal{D}$: finite subset of $\mathcal{H}$): finite subset of $\mathcal{H}$;
begin
    for each $a \in \mathcal{A} \setminus \mathcal{D}$ **do**
    begin
        $\mathcal{C} := \{b \in \mathcal{A} \setminus \{a\}$ such that $a \cap b \neq \emptyset\}$;
        if ($\mathcal{C} \neq \emptyset$) **and** (*reduce* ($\mathcal{C}$, $\emptyset$) has exactly one element) **then**
            return *reduce* ($\mathcal{A} \setminus \{a\}$, $\mathcal{D}$);
    end;
    return $\mathcal{A}$;
end;

Proposition 11 *Given finite sets $\mathcal{A} \subset \mathcal{H}$ and $\mathcal{D} \subset \mathcal{A}$, Algorithm 10 always stops and returns a set of hypercubes $\mathcal{B} \subset \mathcal{A}$ such that $\mathcal{D} \subset \mathcal{B}$ and the inclusion $|\mathcal{B}| \subset |\mathcal{A}|$ induces an isomorphism in homology.*

Proof. The proof proceeds by induction on the cardinality of $\mathcal{A}$. If $\mathcal{A}$ consists of a single hypercube, the conclusion is trivial. Assume the proposition holds true for families $\mathcal{A}$ consisting of less than n elements. Let $\mathcal{A} \subset \mathcal{H}$ be of cardinality n. First observe that all recursive calls of *reduce* complete in finite time by the induction assumption, because in each call the first argument is contained in $\mathcal{A} \setminus \{a\}$. Therefore *reduce* called with the n-element set $\mathcal{A}$ as the first argument will stop. It will either return $\mathcal{A}$ or $\mathcal{A}' :=$ *reduce* $(\mathcal{A} \setminus \{a\}, \mathcal{D})$ for some a for which the corresponding set *reduce* $(\mathcal{C}, \emptyset)$ has exactly one element. In the first case the conclusion is obvious. Hence consider the other case. By the induction assumption $\mathcal{D} \subset \mathcal{A}' \subset \mathcal{A} \setminus \{a\} \subset \mathcal{A}$ and the inclusion $|\mathcal{A}'| \subset |\mathcal{A} \setminus \{a\}|$ induces an isomorphism in homology. For the same reason $|\mathcal{C}|$ has the same homology as $|$*reduce* $(\mathcal{C}, \emptyset)|$, which is a single hypercube. Therefore the homology of $|\mathcal{C}|$ is trivial and by Lemma 9 so is the homology of $a \cap |\mathcal{C}| = a \cap |\mathcal{A} \setminus \{a\}|$. It follows from the Mayer-Vietoris sequence that the inclusion $|\mathcal{A} \setminus \{a\}| \subset |\mathcal{A}|$ induces an isomorphism in homology. Therefore the inclusion $|\mathcal{A}'| \subset |\mathcal{A}|$ induces an isomorphism in homology as a composition of two isomorphisms. The conclusion for $\mathcal{A}$ of cardinality n follows. □

5 Computation of the Conley Index

In this section we describe algorithms which may be used in the proof of the fact that the Conley index of the isolating neighbourhood constructed by Algorithm 7 is the one of an attracting periodic orbit.

First we recall two algorithms, whose bodies and proofs of correctness may be found in [1] and [3] respectively.

Algorithm 12
function *chainmap* ($\mathcal{A}$, $\mathcal{B}$: finite subset of $\mathcal{H}$,
  $\mathcal{F} : \mathcal{A} \to 2^{\mathcal{B}}$: almost perfect multivalued cubical map):
  (C, D: chain complexes of $|\mathcal{A}|$ and $|\mathcal{B}|$,
  $\psi : C \to D$: chain map of $|\mathcal{F}|$);

Algorithm 13
function *homology* (C, D: finitely generated free chain complex,
  $\psi : C \to D$: chain map):
  (HC, HD: homology modules of C and D with coefficients in $\mathbb{Q}$,
  $H\psi : HC \to HD$: homology of the map ψ);

Let us now focus on the following algorithm, which is a combination of the algorithms introduced so far:

Algorithm 14
function *computations* ($\mathcal{N}_1$: finite subset of $\mathcal{H}$,
$\quad\mathcal{F} : \mathcal{A} \to 2^{\mathcal{H}}$: finite multivalued cubical map):
$\quad(HC,\ HD$: homology module, $H\psi : HC \to HD)$;
begin
$\quad\mathcal{N}$:= *neighbourhood* $(\mathcal{N}_1,\ \mathcal{F})$;
$\quad\mathcal{N}'$:= *reduce* $(\mathcal{N},\ \emptyset)$;
$\quad$**if** $\mathcal{F}$ is not almost perfect on $\mathcal{N}'$ **then**
$\qquad$**return** $(\emptyset,\ \emptyset,\ \emptyset)$;
$\quad\mathcal{N}''$:= *reduce* $(\mathcal{N},\ \mathcal{F}(\mathcal{N}'))$;
$\quad(C,\ D,\ \psi)$:= *chainmap* $(\mathcal{N}',\ \mathcal{N}'',\ \mathcal{F})$;
$\quad$return *homology* $(C,\ D,\ \psi)$
end;

Proposition 15 *Let $\mathcal{N}_1$ be a finite subset of $\mathcal{H}$. Let $\mathcal{F}$ be a cubical enclosure of φ_t. If Algorithm 14 stops and returns HC, HD and $H\psi$ such that HC or HD is isomorphic to the homology module of the circle S^1 and $H\psi$ is an isomorphism then $N := |\mathcal{N}|$ has the cohomological Conley index of an attracting periodic orbit with respect to the flow φ.*

Proof. First of all, let us notice that it follows from Proposition 11 that HC and HD are both isomorphic to the homology module of N. Moreover, $H\psi$ corresponds to the homology of $\varphi_t : N \to N$ by Corollary 4.6 and 4.7 in [1]. Since we work with compact cubical sets in $\mathbb{R}^n$ which are obviously polyhedra, it follows that we have the same result for the Alexander-Spanier cohomology, i.e., the cohomology of N is isomorphic to the cohomology of S^1 and the map φ_t on N induces an isomorphism in cohomology. From Proposition 2 it follows that φ_t in N has the cohomological Conley index of an attracting periodic orbit. By definition this is the Conley index of $\mathrm{inv}(N, \varphi_t)$. By Proposition 8 $\mathrm{inv}(N, \varphi) = \mathrm{inv}(N, \varphi_t)$, and it follows from Theorem 5 that the cohomological Conley index of $\mathrm{inv}(N, \varphi)$ is the Conley index of an attracting periodic orbit. $\square$

6 Verification of the Existence of a Poincaré Section

In this section we introduce a method which may be used to prove that an isolating neighbourhood admits a Poincaré section. In the sequel the notation I_K will stand for the set $\{0, 1, \ldots, K\}$.

Algorithm 16
function *verify* ($\mathcal{A}$: finite subset of $\mathcal{H}$, Q: subset of $\mathbb{R}^n$, τ: **real**,
    K: **integer**, F: function $\mathcal{A} \times I_K\tau \to 2^{\mathbb{R}^n}$): **boolean**;
begin
    for each $a \in \mathcal{A}$ **do**
    begin
        answer := **false**;
        for $k := 0$ **to** K **do**
            if $F(a, k\tau) \subset \operatorname{int} Q$ **then**
                answer := **true**;
        if answer = **false then**
            return false
    end;
    return true
end;

Proposition 17 *Let $\mathcal{A}$ be a finite subset of $\mathcal{H}$. Assume $\Xi \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^n$ are such that $\Xi \cap \operatorname{int} Q = \emptyset$ and $\varphi(x, \mathbb{R}^+) \cap \Xi \neq \emptyset$ for each $x \in \operatorname{int} Q$. Let $\tau > 0$, $K \in \mathbb{Z}^+$, and $F : \mathcal{A} \times I_K\tau \to 2^{\mathbb{R}^n}$ be a map such that $\varphi(a, k\tau) \subset F(a, k\tau)$ for each $a \in \mathcal{A}$ and $k \in I_K$. If Algorithm 16 returns* **true** *then $\varphi(x, \mathbb{R}^+) \cap \Xi \neq \emptyset$ for every $x \in |\mathcal{A}|$.*

Proof. Given $x \in |\mathcal{A}|$, there exists $a \in \mathcal{A}$ such that $x \in a$. For this a there exists $k \in I_K$ such that $F(a, k\tau) \subset \operatorname{int} Q$, because otherwise Algorithm 16 returns **false**. Denote $\varphi(x, k\tau)$ by y. Since $y \in \operatorname{int} Q$, there exists $s > 0$ such that $\varphi(y, s) \in \Xi$. Then $\varphi(x, k\tau + s) \in \Xi$, which completes the proof. $\square$

7 Application to the Rössler Equations

In this section we apply our method to give a computer assisted proof of Theorem 1. The results of computations mentioned here are available in [12].

Let φ be the continuous dynamical system generated by the Rössler system (1). Let $\Xi = \{0\}\times[-6,-3]\times[-0.5,0.5]$ and let $Q = [-1,0]\times[-4.5,-3]\times[-0.5,0.5]$.

Lemma 18 *The Rössler vector field is transverse to Ξ, $\Xi \cap \operatorname{int} Q = \emptyset$ and $\varphi(q, \mathbb{R}^+) \cap \Xi \neq \emptyset$ for each $q \in \operatorname{int} Q$.*

Proof. The transversality of the Rössler vector field to Ξ is obvious, as well as the emptiness of the intersection of Ξ with the interior of Q. To see the last property of Q, take an auxiliary set $R = [-1,0]\times[-6,-3]\times[-0.5,0.5]$. Let $q \in \operatorname{int} Q$. We will show that there exists $t \in (0, 0.5)$ such that $\varphi(q,t) \in \Xi$ and $\varphi(q,[0,t]) \subset R$. Let $t = \min\{\tau > 0 \mid \varphi(q,\tau) \in \partial R\}$. Note that $t \in (0, 0.5)$, because $\dot{x} > 2$ in R. Since for $z = -0.5$ we have $\dot{z} > 0$ in R and for $z = 0.5$ we have $\dot{z} < 0$ in R, we can easily see that $\varphi(q,t)$ is neither at the upper nor at the lower face of R. Since $\dot{y} \in [-2.2, -0.6]$ in R, we can see that $\varphi(q,t)$ cannot be in the plane $\{y = -6\}$ and $\{y = -3\}$. Recall that $\dot{x} > 0$ in R to conclude that the only possibility for $\varphi(q,t)$ is to be in the plane $P := \{x = 0\}$. But $\Xi = P \cap R$ and thus $\varphi(q,t) \in \Xi$. $\square$

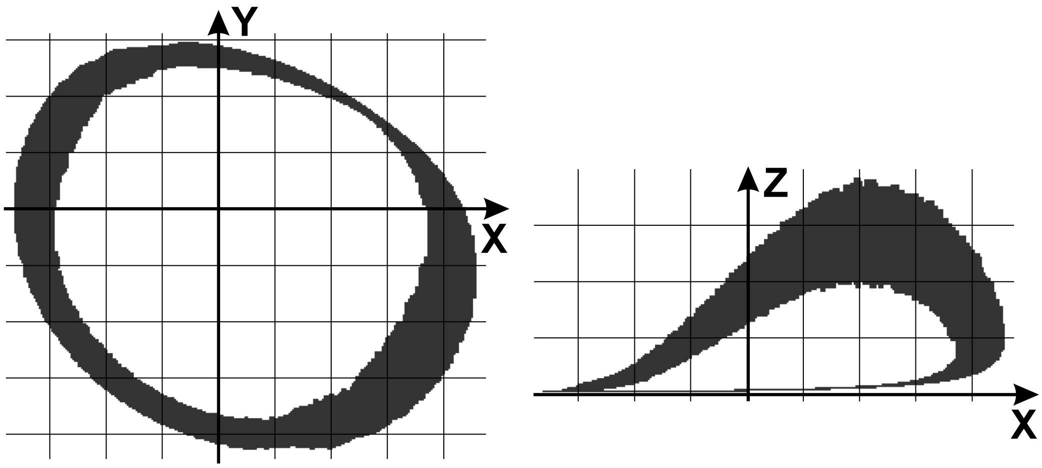

Figure 1: Projections of the neighbourhood N to the planes XY and XZ.

Proof of Theorem 1. Let $d = \frac{1}{32}$. Take $\mathcal{N}_1 = \{[-\frac{107}{32}, -\frac{106}{32}] \times [-\frac{1}{32}, 0] \times [\frac{1}{32}, \frac{2}{32}]\}$. Let $t = 2$. Apply Algorithm 14 to $\mathcal{N}_1$ and an almost perfect cubical enclosure $\mathcal{F}$ of φ_t on $[-6,6]^3$ computed with Lohner's method. It stops and returns HC, HD and $H\psi$ such that HC and HD are isomorphic to the homology of S^1 and $H\psi$ is an isomorphism. By Proposition 15 the constructed isolating neighbourhood $N = |\mathcal{N}|$ has the cohomological Conley index of an attracting periodic orbit. Algorithm 16 applied to $\mathcal{N}$, Q, $\tau := \frac{1}{16}$, $K := 120$ and F computed again with Lohner's method returns **true**. It follows from Proposition 17 that N admits a Poincaré section. Thus the assumptions of Theorem 3 are satisfied. It follows that the set N contains a periodic orbit. □

8 Remarks and Comments

In this section some technical remarks on various aspects of the computations are gathered, and the cost of implementation is roughly discussed.

The most important parameters in the construction of an isolating neighbourhood are grid size $d > 0$ and time $t > 0$. They are vital for Algorithm 7, which is the most time consuming part of the proof and the remaining computations depend on its result. Before proceeding with this algorithm, some preliminary simulations may help make an appropriate choice of d and t. Instead of using Lohner's algorithm, the image of each hypercube may be computed by integrating its vertices with an approximate numerical method (like the Runge-Kutta or Euler methods) and by taking a minimal convex cubical set containing them. Since Lohner's method implemented by Mrozek and Zgliczyński produces very tight enclosures, in most cases the rigorously computed set N should not be significantly larger than the estimated one.

grid size	number of cubes	volume of the neighbourhood	computation time on a PC 133 MHz	time of reduction to $\mathcal{N}'$
1/16	failure			
1/32	159,677	4.87	1.5 days	33 minutes
1/64	293,756	1.12	2.7 days	57 minutes
1/128	563,364	0.27	5.2 days	99 minutes
1/256	1,097,512	0.065	10 days	170 minutes

Table 1: Different isolating neighbourhoods for different grid sizes.

In Table 1 the size of the neighbourhood and the cost of its construction in the case of the Rössler system (1) is illustrated as a function of the grid size $d > 0$. It is worth to point out that the volume of the neighbourhood, which may serve as a measure of precision in locating the periodic orbit, decreases rapidly with the grid size. In the case of $d = 1/16$ the algorithm fails in creating a suitable neighbourhood because of producing too large enclosures of images of some of the cubes, which is caused by the lack of reasonable estimations for the vector field over large areas.

time t	number of cubes	volume of the neighbourhood	computation time on a PC 133 MHz
0.25	f a i l u r e		
0.5	1,063,211	32.4	2.5 days
1.0	376,171	11.5	1.7 days
2.0	159,677	4.87	1.5 days
3.0	66,521	2.03	22 hours
4.0	53,725	1.64	24 hours
5.0	41,034	1.25	23 hours
6.0	f a i l u r e		

Table 2: Different isolating neighbourhoods for different times.

In Table 2 several choices of $t > 0$ are listed together with some features of the neighbourhoods constructed for them by Algorithm 7. If the cubes have little time to be attracted towards the periodic trajectory, the algorithm constructs a large neighbourhood (like the one for $t = 0.5$) or seems to loop infinitely (as for $t = 0.25$). On the other hand, if the cubes are iterated for a long period of time, they grow to such an extent that applying this method to them becomes practically impossible, which is the case of $t \geq 6$.

If the set $\mathcal{N}$ is small, Algorithms 12 and 13 may be run without prior computation of $\mathcal{N}'$ and $\mathcal{N}''$ with Algorithm 10. However, this may only be the case of simple examples, because the algebraical structures being created during the computation of the homology of N and φ_t are very large (cf. [1, 3, 5]).

The set $\mathcal{N}_1$ is meant to be an initial approximation of the orbit, but in most cases it is sufficient to take a single hypercube close to the orbit and let it propagate into a full neighbourhood. This was the case of the proof of Theorem 1. However, an appropriate set $\mathcal{N}_1$ may be a good hint for reduction of $\mathcal{N}$ to $\mathcal{N}'$ by Algorithm 10.

Finding sets satisfying the assumptions of Proposition 17, like it was illustrated in Lemma 18, is usually very simple and involves elementary analysis of the vector field f. It is important to find a possibly large set Q in order to allow producing loose enclosures of φ by the function F in Algorithm 16, which allows these computations to complete significantly faster than during the construction of N. Moreover, if the grid size d is very small, in this part of computations the set $\mathcal{N}$ may be replaced by a cubical set $\mathcal{N}^+$ with respect to a larger grid, provided $|\mathcal{N}| \subset |\mathcal{N}^+|$.

We would like to emphasize that in this method an isolating neighbourhood containing the periodic trajectory is constructed. This is in contrast with the Poincaré map methods which produce at most a subset of the Poincaré section containing its intersection with the periodic trajectory.

This method may be generalized to detect hyperbolic periodic orbits which are not necessarily attracting. For this purpose, an isolating neighbourhood N should be determined *a priori*, and an algorithm for construction of an index pair in N must be used (an example may be found in [11]). Moreover, an algorithm for computation of relative homology of this pair of cubical sets must be used, together with computation of homology of the index map. Such algorithms are currently under development and should be available soon.

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