Representing double-layer cubical sets with (ordinary) cubical sets in $\mathbb{R}^{n+1}$

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Abstract

In this note which accompanies the paper *Excision-preserving cubical approach to the algorithmic computation of the discrete Conley index* the details of representing double-layer cubical sets in terms of (ordinary) cubical sets for the purpose of homology computation are explained.

1 Introduction

In the paper [4], a double-layer cubical grid structure with identification on some subset is introduced. Although the modification of various algorithms dealing with (ordinary) cubical sets necessary to deal with the new objects is not very complicated, as one can see in the CHomP software package [1] (programmed mainly in the file `include/cubes/twolayer.h`), for some purposes it might be desirable to avoid these modifications and to work directly with (plain) cubical sets instead.

In this note a combinatorial index pair $(\mathcal{P}_1, \mathcal{P}_2)$ built of full cubes in $\mathbb{R}^n$ and a combinatorial map $\mathcal{F}$ on the index pair are considered as the input data, and it is explained how to translate this data into full cubical sets in $\mathbb{R}^{n+1}$ in such a way that the topology of the double-layer construction introduced in [4] is achieved (up to homotopy), and therefore, the result of homology computation is the same as in the case of the double-layer approach.

2 Embedding

Recall from [4] that the index pair $(\mathcal{P}_1, \mathcal{P}_2)$ and its image $\big(\mathcal{P}_1 \cup \mathcal{F}(\mathcal{P}_2), \mathcal{P}_2 \cup \mathcal{F}(\mathcal{P}_2)\big)$ are lifted into a double-layer space by adding layer information into each full cube, and thus replacing Q by $[Q, i]$, where $i \in \{0, 1\}$. Please, see [4] for the topology introduced on this space.

For the purpose of homology computation, the cubes in $\mathcal{P}_1 \setminus \mathcal{P}_2$ as well as their images by $\mathcal{F}$ are put at level 1, whereas the cubes in $\mathcal{P}_2$ and their images by $\mathcal{F}$ are set at level 0. Moreover, there is an identification of the layers introduced on $\mathcal{P}_2$, that is, $[Q, 0] \simeq [Q, 1]$ iff $Q \in \mathcal{P}_2$. These cubes with the level information are embedded into $\mathbb{R}^{n+1}$ as full cubes as follows:

$$[Q, i] \mapsto \begin{cases} Q \times [0, 1] & \text{if } Q \notin \mathcal{P}_2 \text{ and } i = 0 \\ Q \times [1, 2] & \text{if } Q \in \mathcal{P}_2 \\ Q \times [2, 3] & \text{if } Q \notin \mathcal{P}_2 \text{ and } i = 1 \end{cases}$$

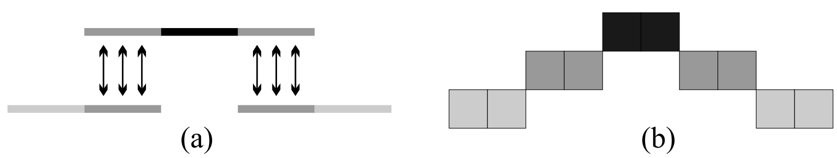

Figure 1: Illustration of an embedding in $\mathbb{R}^2$ of the Ω-space from Fig. 2 in [4].

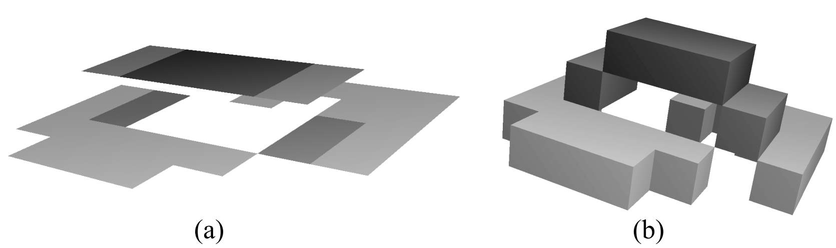

Figure 2: Embedding of the index pair shown in Fig. 1 in [4].

The Figures 1 and 2 illustrate this embedding. In the left-hand side picture (a), the set $|\mathcal{P}_1 \setminus \mathcal{P}_2| \times \{1\}$ is indicated in dark grey, $|\mathcal{P}_2| \times \{0, 1\}$ is painted in medium grey, and $|\mathcal{F}(\mathcal{P}_2) \setminus \mathcal{P}_2| \times \{0\}$ is visualized in light grey. In the right-hand side picture (b), the set $|\mathcal{P}_1 \setminus \mathcal{P}_2| \times [2, 3]$ is shown in dark grey, $|\mathcal{P}_2| \times [1, 2]$ is drawn in medium grey, and $|\mathcal{F}(\mathcal{P}_2) \setminus \mathcal{P}_2| \times [0, 1]$ is plotted in light grey.

3 Software

The translation of an n-dimensional combinatorial index pair and the accompanying combinatorial map into the $(n+1)$-dimensional setting described above has been programmed in the Perl script `liftcubes.pl` available in the software package [1].

After this translation, the homology computation can be carried out by the old program `homcubes` which follows the algorithms introduced in [2], and thus there is no need to use the program `homcub2l` which works on the double-layer data structures. Both programs for the homology computation are also available in [1].

4 Conclusion

Although the approach introduced in this note eliminates any potential difficulties which might arise from the fact of using the double-layer cubical grid structure, its importance turns out to be not good enough to be included in our paper [4], mainly because it gives rise to a substantial slow-down in the computations due to the increase in the dimension of cubes, as one can see in Table 1 in [4]. Nevertheless, this approach is considered worth being published at the website [3] containing the data referred to in that paper, and this is the reason for the existence of this note.

References

[1] *The CHomP Software*, in: *Computational Homology Project* (1997-), `http://chomp.rutgers.edu/`.

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[3] P. Pilarczyk, *Excision-preserving cubical approach to the algorithmic computation of the discrete Conley index. Software and examples*, `http://www.pawelpilarczyk.com/excision/`

[4] P. Pilarczyk, K. Stolot, *Excision-preserving cubical approach to the algorithmic computation of the discrete Conley index*, submitted.