Dynamics of homeomorphisms of surfaces

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The dynamics of a self-map $f : X \to X$ is described by invariants as the sequence of numbers of periodic points $\{|P^n(f)|\}$, $P^n(f) = \mathrm{Fix}(f^n)$, the set of minimal periods $Per(f) \in \mathbb{N}$, or the topological entropy $\mathbf{h}(f) \in \mathbb{R}^+ \cup \{0\}$. In the case when $X = S_{\mathrm{g}}$ is a an oriented surface of genus g, $\mathrm{g} \geq 2$, and f is a homeomorphism we can use many topological tools due to the Nielsen-Thurston classification theorem. It states that up to a homotopy, thus isotopy, every homeomorphism f of S_{g} is represented by: either a periodic homeomorphism, or a pseudo-Anosov homeomorphism, or a reducible homeomorphism. In the last case each piece of the reduction can be periodic, or pseudo-Anosov. It is important, but difficult, to distinguish when f is the pseudo-Anosov or its reduction contains at least one pseudo-Anosov piece, because then $Per(f)$ is infinite, e.g. there is infinitely many periodic points, and $\mathbf{h}(f) = \log(\lambda_f)$ where λ_f is the largest stretching (expanding) factor of the pseudo-Anosov piece. From the Nielsen theory we know that $\lambda_f = N^\infty(f) := \limsup \sqrt[n]{N(f^n)}$, but an affective computation of the latter is also strenuous. In last two decades, to resolve this difficulty there was developed a theory which let us to express λ_f as the as the spectral radius of $H_1(\tilde{f})$, where $H_1(\tilde{f}) \in Sp(2\mathrm{g}, \mathbb{Z})$ is the linear map induced on $H_1(\tilde{S}_{\tilde{\mathrm{g}}}; \mathbb{Z}) = \mathbb{Z}^{2\tilde{\mathrm{g}}}$ by a homeomorphism $\tilde{f} : \tilde{S}_{\tilde{\mathrm{g}}} \to \tilde{S}_{\tilde{\mathrm{g}}}$ being a lift of f to a finite regular cover $p : \tilde{S}_{\tilde{\mathrm{g}}} \to S_{\mathrm{g}}$. On the other hand, the results of 70ties of 20th century showed that the dynamics of a pseudo-Anosov homeomorphism of surface, e.g. its entropy, can be represented as the dynamics of quotient of finite sub-shift (a Markov partition). In this talk we pose a question whether a combination of two mentioned approaches could lead to an effective, computer assisted, method of distinguish whether there is a pseudo-Anosov piece in the canonical form of f and which let us to derive λ_f.

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