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TOPOLOGICAL-NUMERICAL APPROACH TO THE EXISTENCE OF PERIODIC TRAJECTORIES IN ODE'S

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Abstract. We discuss an application of a topological-numerical method for proving the existence of a periodic trajectory in a smooth dynamical system in $\mathbb{R}^n$ where a periodic trajectory is numerically observed. The method is based on the Conley index theory and rigorous numerics for ODEs and it is a generalization of the method introduced in [13]. We apply this method to the Rössler equations.

1. **Introduction.** The aim of the paper is to discuss a specific application of a recently developed topological-numerical method for proving the existence of a periodic trajectory in a smooth dynamical system in $\mathbb{R}^n$ where a periodic orbit is numerically observed. The method is based on the Conley index theory [3, 8] and rigorous numerics for ODEs [6, 12, 21] and it is a generalization of the method introduced in [13, 14]. We discuss an application of this method to the Rössler equations and we prove the following

Theorem 1. *The Rössler system* [18]

$$\begin{aligned} \dot{x} &= -(y+z), \\ \dot{y} &= x + by, \\ \dot{z} &= b + z(x-a) \end{aligned} \tag{1}$$

for $a = 3.1$ *and* $b = 0.2$ *admits two periodic orbits.*

The existence of one of the orbits was proved in [11]. This is the stable periodic orbit that emerges in the period-doubling bifurcation observed in numerical simulations when the parameter a is increased from 2.2 to 3.1. However, with the increase of the value of a, the stable periodic trajectory which exists for $a = 2.2$ (proved in [13]) becomes unstable and the method used in [13] does not allow one to prove its existence. This problem is addressed in this paper.

The existence of the periodic orbits considered in Theorem 1 was conjectured by Hale and Koçak in [4], where also numerical evidence of the existence of the periodic orbits in this system was given.

Needless to say, periodic solutions to ordinary differential equations in $\mathbb{R}^n$ are among the basic objects of interest in the theory of dynamical systems. Because

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of this, developing an easy-to-apply method for proving their existence in concrete dynamical systems is of significant importance.

As indicated in [13], our method has some advantages over the methods based on the analysis of the Poincaré map. The applications discussed in [11, 13] prove its usefulness. However, the method from [13] can only be applied to stable periodic trajectories. In this paper we discuss its generalization which may be applied to virtually any autonomous differential equation in $\mathbb{R}^n$ which exhibits in numerical simulations a hyperbolic periodic trajectory. Due to substantial difference in complexity of the problem considered here in comparison to [13], new algorithms and techniques had to be developed to address this problem.

It needs to be mentioned that our approach is constructive in the sense that the periodic orbit is proved to be in an effectively constructed neighborhood of the numerically observed trajectory. This neighborhood is obtained as a set built of (hyper)cubes and may be a subject of further analysis or visualization.

Moreover, one should notice that the combinatorial procedure for finding an index pair described in Section 3 may have more applications than the one discussed in this paper and in some situations may be a good alternative to the procedure introduced in [20].

2. **Preliminaries.** Consider the differential equation

$$\dot{x} = f(x), \tag{2}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a vector field of class C^1. Such a vector field induces a dynamical system φ on $\mathbb{R}^n$ under some additional assumptions, for example, if f is bounded. For the purpose of this paper, we can assume this without loss of generality, because our method is local, that is, we can restrict our attention to a bounded region in $\mathbb{R}^n$ within which we look for a periodic trajectory, and we do not even need to know f outside this region. Therefore, if (2) does not induce a dynamical system on $\mathbb{R}^n$, we can modify f in such a way that $f(x) = 0$ if $\|x\|$ is large enough, for instance, beyond the range of representable real numbers we use in numerical computations.

We follow the terminology and notation of [13]. In particular, we work with the time-t discretization φ_t of the flow φ with some fixed $t > 0$. As justified in [13] (see also [10]), if we use this discretization φ_t to find an isolating neighborhood N and to compute the Conley index of its invariant part S, then the result is also valid for the flow, that is, N is an isolating neighborhood with respect to the flow, S is its invariant part with respect to the flow, and the Conley index of S with respect to the flow coincides with the one computed for φ_t.

To prove the existence of a periodic orbit we verify the assumptions of the following theorem, which is a special case of Corollary 1.4 in [8]:

Theorem 2. *Assume N is an isolating neighborhood for the flow φ which admits a Poincaré section Ξ. If N has the cohomological Conley index of a hyperbolic periodic orbit, then* $\operatorname{inv}(N, \varphi)$ *contains a periodic orbit.*

Unfortunately, the definition of an index pair proposed in [8] is too restrictive for our purpose. Therefore, we use the following definition introduced by Szymczak in [19], where it is also proved that this index pair can be used to compute the Conley index as in [8].

Definition 1. A pair of compact sets (P_1, P_0) is called an *index pair* for an isolated invariant set S with respect to a continuous map f if

(1) $\mathrm{cl}(P_1 \setminus P_0)$ is an isolating neighborhood for S,
(2) if $x \in P_0$, then $f(x) \notin P_1 \setminus P_0$,
(3) if $x \in P_1$ and $f(x) \notin P_1$, then $x \in P_0$.

3. **Construction of an Index Pair.** The algorithm introduced in [13] is capable of constructing an index pair (P_1, P_0) only if the exit set P_0 is empty, and therefore cannot be used here. Although the algorithm introduced in [20] does not have this limitation, it is difficult to apply here, because it requires an apriori bound for the region containing the index pair to be constructed. Therefore, we use another algorithm for the construction of an index pair. The substantial difference between this algorithm and the one introduced in [20] is that the latter constructs an index pair *contained in* a given set, and thus requires the computation of the map $\mathcal{F}$ on the entire initial set in which the index pair is going to be contained, which may be very costly due to the huge size of that set, whereas our new algorithm constructs an index pair *containing* a given initial set, and therefore requires the computation of $\mathcal{F}$ only on the constructed sets.

Recall from [13] that

$$\mathcal{H} = \{[k_1 d, (k_1+1)d] \times \cdots \times [k_n d, (k_n+1)d] \mid k_i \in \mathbb{Z} \text{ for } i = 1, \ldots, n\},$$

where $d > 0$ is fixed, and that a map $\mathcal{F}\colon \mathcal{A} \to 2^{\mathcal{H}}$ is called *finite* if $\mathcal{A}$ is finite and $\mathcal{F}(a)$ is a finite subset of $\mathcal{H}$ for each $a \in \mathcal{A}$.

Algorithm 1.
function *IndexPair* ($\mathcal{Q}_1$, $\mathcal{Q}_0$: finite subset of $\mathcal{H}$, $\mathcal{F}\colon \mathcal{A} \to 2^{\mathcal{H}}$: finite
multivalued cubical map): pair of finite subsets of $\mathcal{H}$;
begin
$\mathcal{Q}_0 := \mathcal{Q}_0 \cup (\mathcal{F}(\mathcal{Q}_1) \setminus \mathcal{Q}_1)$;
if $\mathcal{Q}_0 \not\subset \mathcal{A}$ **then return** $(\emptyset, \emptyset)$;
while $\mathcal{F}(\mathcal{Q}_0) \cap \mathcal{Q}_1 \neq \emptyset$ **do**
begin
choose any $a \in \mathcal{Q}_0$ such that $\mathcal{F}(a) \cap \mathcal{Q}_1 \neq \emptyset$;
if $\mathcal{F}(a) \not\subset \mathcal{A}$ **then return** $(\emptyset, \emptyset)$;
$\mathcal{Q}_0 := \mathcal{Q}_0 \cup (\mathcal{F}(a) \setminus \mathcal{Q}_1)$;
$\mathcal{Q}_1 := \mathcal{Q}_1 \cup \{a\}$;
$\mathcal{Q}_0 := \mathcal{Q}_0 \setminus \{a\}$;
end;
return $(\mathcal{Q}_1, \mathcal{Q}_0)$
end.

Note that the algorithm should be invoked with $\mathcal{Q}_1 \subset \mathcal{A}$, or otherwise in the first line $\mathcal{F}(\mathcal{Q}_1)$ is not defined.

Recall that the set represented by $\mathcal{A} \subset \mathcal{H}$ is denoted by $|\mathcal{A}| := \bigcup_{a \in \mathcal{A}} a \subset \mathbb{R}^n$. A finite multivalued cubical map $\mathcal{F}\colon \mathcal{A} \to 2^{\mathcal{H}}$ is called en *enclosure* of a continuous map $f\colon |\mathcal{A}| \to \mathbb{R}^n$ if $f(a) \subset \mathrm{int}|\mathcal{F}(a)|$ for every $a \in \mathcal{A}$.

Theorem 3. *Let $\mathcal{F}\colon \mathcal{A} \to 2^{\mathcal{H}}$ be an enclosure of a continuous map $f\colon \mathbb{R}^n \to \mathbb{R}^n$, and let $(\mathcal{Q}_1^0, \mathcal{Q}_0^0)$ be a pair of disjoint subsets of $\mathcal{A}$. Denote by $(\mathcal{Q}_1, \mathcal{Q}_0)$ the pair returned by Algorithm 1 invoked with $\mathcal{Q}_1^0$, $\mathcal{Q}_0^0$ and $\mathcal{F}$. Then the pair $(P_1, P_0) := (|Q_1 \cup Q_0|, |Q_0|)$ is an index pair for some isolated invariant set S with respect to f and $|\mathcal{Q}_1|$ is an isolating neighborhood for S.*

For proof of Theorem 3 the reader is kindly requested to consult [16].

4. **Construction of the Proof.** In this section we describe how we verify the assumptions of Theorem 2 in order to prove that a given ODE admits a periodic trajectory.

Before we proceed, let us notice that the Conley index of the constructed index pair (P_1, P_0) cannot be computed with the algorithms that were used in [13]. The first reason for this is the fact that—in contrast to the situation considered in [13]—the exit set P_0 can be nonempty, and therefore the map induced by f in relative homology of the pair (P_1, P_0) must be computed. In [13] an algorithm from [1] (see also [7]) was used to construct a chain map suitable for homology computation, but the authors of [1] do not address the issue of relative homology computation. Moreover, the reduction of the cubical sets ([13], §4), which was a crucial step in decreasing the size of data to process, needs to be improved to handle pairs of sets and relative homology. In addition to that, in our case the data for the computation of the map induced in homology usually turns out to be too large to be processed by the algorithm introduced in [1] in reasonable time, even after the reduction of the cubical sets.

The above-mentioned issues were among the motivations for the development of a new algorithm for homology computation of acyclic multivalued cubical maps, which is capable of handling relative homology, contains an improved version of geometric reduction of pairs of cubical sets analogous to the one introduced in [13], and uses an approach substantially different from the one suggested in [1] for the homology computation of maps, which makes it much more efficient. This algorithm has been developed recently and is described in [9] (see also [15]).

The proof of the existence of a periodic trajectory begins with the construction of an index pair using Algorithm 1 for an initial set $\mathcal{Q}_1^0$ taken as a rough approximation of the periodic trajectory whose existence we expect to prove. The set $\mathcal{Q}_1^0$ can be obtained in numerical experiments.

We compute an enclosure $\mathcal{F}$ (to use in Algorithm 1) for the time-t map φ_t for the flow φ generated by the ODE of interest with the software package `capd` [2] which uses advanced numerical methods to provide rigorous bounds for images of cubes under the map φ_t (see [6, 12, 21]).

In the next step we compute the Conley index of the constructed index pair.

Finally, the verification of the existence of a Poincaré section, which is one of the assumptions of Theorem 2, can be performed in the same way as described in [13].

In a more detailed way, the method for proving the existence of a periodic trajectory in a given continuous dynamical system consists of the following steps:

1. Fix $d > 0$ and find a finite set $\mathcal{Q}_1^0 \subset \mathcal{H}$ that roughly approximates the trajectory of interest.
2. Fix a finite set $\mathcal{A} \subset \mathcal{H}$ within which one expects to construct the index pair.
3. Fix $t > 0$ and run Algorithm 1 with $\mathcal{Q}_1^0$, $\mathcal{Q}_0^0 := \emptyset$ and an enclosure $\mathcal{F}$ of φ_t.
4. Compute the Conley index of the index pair with respect to φ_t.
5. Prove that the isolating neighborhood $|\mathcal{Q}_1|$ admits a Poincaré section.

Note that in Step 3 the map $\mathcal{F}$ does not have to be computed on the entire set $\mathcal{A}$. One can program a subroutine that is invoked for each cube $a \in \mathcal{A}$ when its image needs to be known, that is, when $a \in \mathcal{Q}_1 \cup \mathcal{Q}_0$. Therefore, taking $\mathcal{A}$ as a huge set containing our region of interest does not increase the complexity of computations, but may be very useful if the desired size of the index pair is not known.

Moreover, in Step 4 the map $\mathcal{F}$ already computed in Step 3 can often be used to compute the index map, which can save a significant amount of computations.

5. **Application to the Van der Pol equations.** Let us show an application of the described method to an easy-to-illustrate planar example. Consider the Van der Pol equations in the form of an autonomous ODE as discussed in [5]. This ODE has a stable periodic trajectory for $a = 1.0$ (a limit cycle), so let us reverse the time to make this trajectory unstable:

$$\begin{aligned} \dot{x} &= -y + x(x^2 - a), \\ \dot{y} &= x. \end{aligned} \tag{3}$$

Note that the existence of the periodic trajectory here for $a = 1.0$ can be shown in a much more direct way; however, we consider this ODE as a straightforward illustration of the way our method is applied.

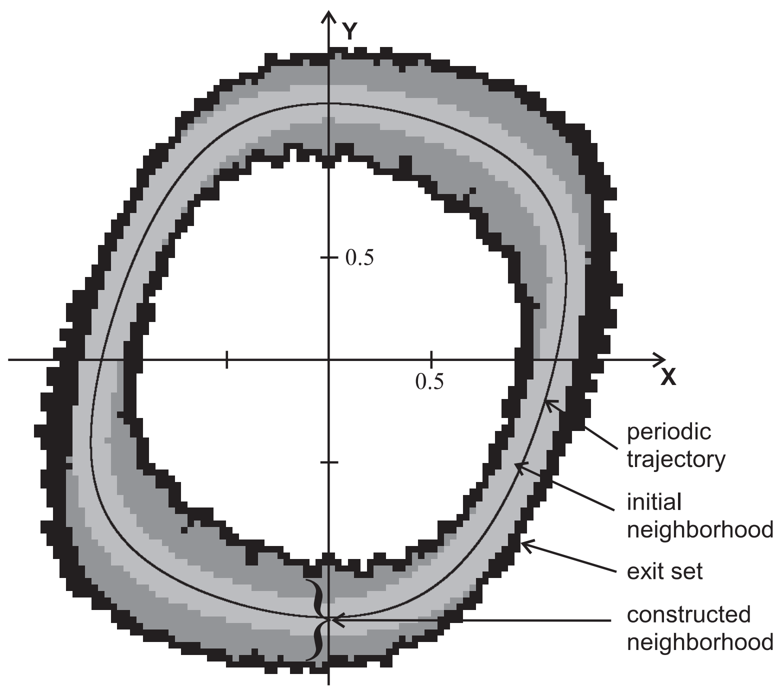

FIGURE 1. Sets constructed for the Van der Pol equations with reversed time.

Let $d := \frac{1}{32}$ and $t := \frac{1}{4}$. Let $A := [-2, 2]^2$ and take $\mathcal{A} := \{a \in \mathcal{H} \mid a \subset A\}$. We run Algorithm 1 with $\mathcal{Q}_1^0$ shown in Figure 1 in light gray, $\mathcal{Q}_0^0 := \emptyset$ and the map $\mathcal{F}$ computed with the `capd` package [2] as explained in Section 4. The obtained set $\mathcal{Q}_1$ is indicated in Figure 1 as the union of the two sets in two shades of gray (note that $\mathcal{Q}_1^0 \subset \mathcal{Q}_1$), and $\mathcal{Q}_0$ is indicated in black. The results of the homology computation of the index pair are as follows:

$$\begin{aligned} H_0(P_1, P_0) &\cong 0, \\ H_1(P_1, P_0) &\cong \mathbb{Z}, \\ H_2(P_1, P_0) &\cong \mathbb{Z}. \end{aligned}$$

The computation of the maps in homology proves that the inverse of the isomorphism induced in homology by the inclusion $i\colon (P_1, P_0) \to (P_1 \cup \varphi_t(P_0), P_0 \cup \varphi_t(P_0))$

composed with the map induced by $\varphi_t\colon (P_1, P_0) \to (P_1 \cup \varphi_t(P_0), P_0 \cup \varphi_t(P_0))$ is the identity. Therefore, the Conley index of the index pair (P_1, P_0) is the index of a hyperbolic periodic trajectory.

We use the technique introduced in [13] to show that the constructed isolating neighborhood admits a Poincaré section. As a result we obtain the following

Theorem 4. *The system (3) for $a = 1$ admits a periodic trajectory.*

6. **Application to the Rössler Equations.** In this section we discuss an application of our method to the Rössler equations (1) and we give a computer assisted proof of Theorem 1. The results of computations mentioned here are available in [17].

Define $\Xi := \{0\} \times [-6, 3] \times [-1/2, 1/2]$ and $Q := [-1, 0] \times [-9/2, -3] \times [-1/2, 1/2]$. One can use the elementary technique from the proof of Lemma 18 in [13] to prove the following

Lemma 1. *The Rössler vector field (1) is transverse to Ξ, and $\varphi(q, \mathbb{R}^+) \cap \Xi \neq \emptyset$ for each $q \in \operatorname{int} Q$.*

Proof of Theorem 1. The existence of one periodic trajectory was already proved in [11]. We will prove the existence of another one. Let $d := \frac{1}{128}$. Take $\mathcal{Q}_1^0$ to be the smallest set of cubes in $\mathcal{H}$ which covers an approximation of the unstable periodic trajectory observed in numerical simulations. Let $A := [-8, 8]^2$ and define $\mathcal{A} := \{a \in \mathcal{H} \mid a \subset A\}$. Run Algorithm 1 with $\mathcal{Q}_1^0$, $\mathcal{Q}_0^0 := \emptyset$ and the map $\mathcal{F}$ computed with the `capd` package [2] as explained in Section 4. The neighborhood $N := |\mathcal{Q}_1|$ and the set $\mathcal{Q}_0$ returned by this algorithm are illustrated in Figures 2 and 3, respectively.

The Conley index of the pair (P_1, P_0) computed with the algorithms introduced in [9] is the index of a hyperbolic periodic trajectory.

We follow the way of reasoning from [13] (Section 7) and we use the sets Ξ and Q which appear in Lemma 1 to prove that N admits a Poincaré section.

In this way we have verified that the assumptions of Theorem 2 are satisfied. It follows that the set N contains a periodic orbit.

Since N and the neighborhood obtained in [11] are disjoint, the periodic trajectory whose existence has just been proven is different from the one considered in [11]. Therefore, the Rössler system (1) indeed admits two periodic solutions. ☐

7. **Remarks and Comments.** In this section some technical remarks on various aspects of the computations are gathered.

First, let us point out that the procedure introduced in Section 4 can fail in several places. We discuss here two cases in which this can happen and we suggest possible ways of correcting this situation.

In Step 3, if Algorithm 1 returns $(\emptyset, \emptyset)$, then this means that the algorithm tries to construct an index pair larger than we expect. To fix this problem one can choose different d, t and/or $\mathcal{A}$ and try running Algorithm 1 again.

In Step 4, if the computed index is different from the index we expect, then probably the isolating neighborhood is too large and contains much more than we expect. In this case it is recommended to decrease d and/or change t and try the entire procedure again. Actually, this situation happens if we take $d = \frac{1}{64}$ in Section 6: The isolating neighborhood $N := |\mathcal{Q}_1|$ constructed for this value of d contains not only the unstable trajectory, but also the stable one. This problem

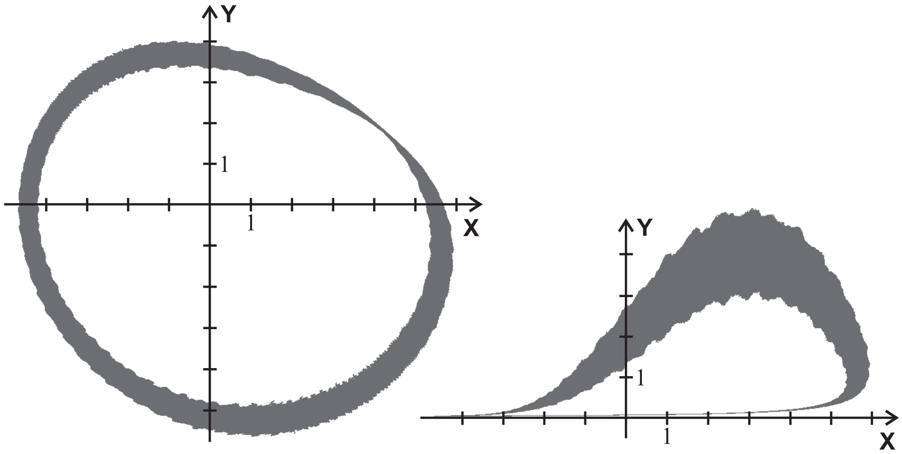

FIGURE 2. Projections of the set $|\mathcal{Q}_1|$ to the planes XY and XZ.

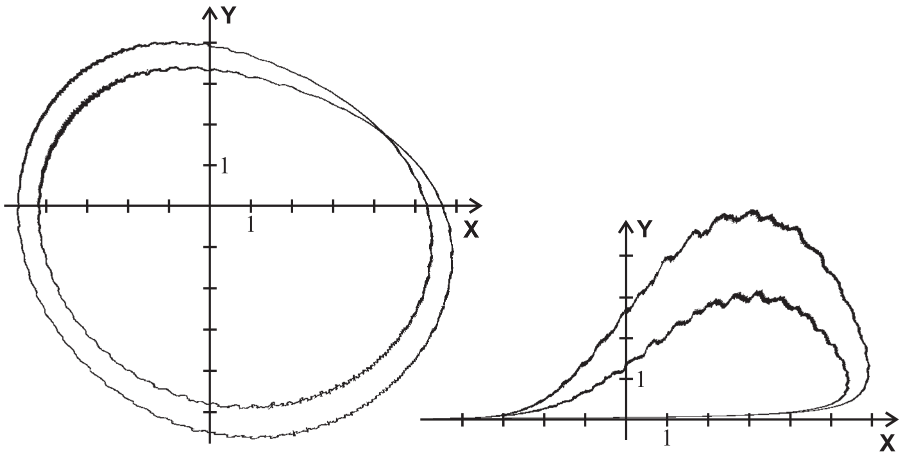

FIGURE 3. Projections of the set $|\mathcal{Q}_0|$ to the planes XY and XZ.

was fixed by decreasing d to $\frac{1}{128}$. It is worth to mention here that one can use N to prove that there exists one more bounded trajectory in addition to the two periodic trajectories. This is due to the fact that the Conley index of N turns out to be different from the direct sum of the Conley indices of the periodic trajectories.

Moreover, it is sometimes worth to adjust the multivalued cubical map which we want to use for homology computation, as well as the index pair obtained by Algorithm 1, in order to speed up the homology computation with the algorithm introduced in [9]. This is one of the reasons why the initial neighborhood for the periodic trajectory considered in Section 5 was taken much larger than the minimal subset of $\mathcal{H}$ which covers the orbit observed in numerical situations (see Figure 1).

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