The Conley Index and Rigorous Numerics of Attracting Periodic Orbits

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1 Introduction

Despite the enormous number of papers devoted to the problem of the existence of periodic trajectories of differential equations, the theory is still far from being satisfactory, especially when concrete differential equations are concerned, because the necessary conditions formulated in many theoretical criteria are difficult to verify in a concrete case. And even if some methods work for some concrete equations, it is usually difficult to carry them over to other problems. Thus quite often the only available method is to experiment numerically. Unfortunately, such an approach cannot be treated as reliable.

All this makes the problem a natural field of research in rigorous numerics. However, only recently some new techniques were developed, for which the amount of computations necessary is in the reach of present-day computers (see [5, 6, 16]). Especially powerful seem to be methods based on topological invariants like the Conley index [5] and the fixed point index [16].

In this paper we want to sketch an approach to the existence of periodic solutions of differential equations based on the discrete Conley index and rigorous numerics of dynamical systems. For details the reader is referred to

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[12, 13]. We briefly discuss the result of applying this method to two different periodic orbits in the Rössler equations and two periodic trajectories in the Lorenz equations.

2 Representable sets and maps

Let X, Y be locally compact metric spaces. For $\mathcal{G} \subset \mathcal{P}(X)$, $A \subset X$ and $\mathcal{C} \subset \mathcal{G}$ put

$$\begin{aligned} \mathcal{G}(A) &:= \{ a \in \mathcal{G} \mid a \cap A \neq \emptyset \}, \\ |\mathcal{G}| &:= \bigcup \mathcal{G}, \\ \langle \mathcal{C} \rangle &:= \langle \mathcal{C} \rangle_{\mathcal{G}} := \{ x \in X \mid \mathcal{G}(x) = \mathcal{C} \}. \end{aligned}$$

A family $\mathcal{G} \subset \mathcal{P}(X)$ will be called a *grid* in X if

(i) every element of $\mathcal{G}$ is a non-empty compact set,

(ii) for every compact $K \subset X$ we have $1 \leq \operatorname{card} \mathcal{G}(K) < \infty$,

(iii) for every $\mathcal{C} \subset \mathcal{G}$ we have $\operatorname{cl} \langle \mathcal{C} \rangle = \bigcap \mathcal{C}$.

A typical example of a grid in $\mathbb{R}^d$ is a set of d-dimensional hypercubes of the same size $\eta > 0$ which fill the space:

$$\mathcal{G}_\eta := \{ \prod_{i=1}^{d} [k_i \eta, (k_i + 1)\eta] \mid k_i \in \mathbb{Z},\ i = 1, \ldots, d \}.$$

Define the *diameter* of a grid $\mathcal{G}$ as

$$\operatorname{diam} \mathcal{G} := \sup \{ \operatorname{diam} a \mid a \in \mathcal{G} \}.$$

A set E is called an *elementary representable set* if $E = \langle \mathcal{C} \rangle$ for a finite subfamily $\mathcal{C} \subset \mathcal{G}$. A set A is called *representable* if it is a finite union of elementary representable sets. A set A is called *strongly representable* if it is a finite union of a subfamily of $\mathcal{G}$.

Theorem 2.1 *(see [10]) The family of representable sets is closed under the set-theoretical union, intersection, difference as well as topological closure and topological interior.*

The family of elementary representable sets over a grid $\mathcal{G}$ will be further denoted by $\mathrm{ER}(\mathcal{G})$, and the family of all representable sets by $\mathrm{R}(\mathcal{G})$.

A *multivalued map* $F\colon X \rightrightarrows Y$ is a map $F\colon X \to \mathcal{P}(Y)$. Its *domain* and *image* are defined as follows:

$$\operatorname{dom} F := \{\, x \in X \mid F(x) \neq \emptyset \,\},$$
$$\operatorname{im} F := \bigcup_{x \in X} F(x).$$

The *image* and *preimage* of a set under a multivalued map is defined in the following way:

$$F(A) := \bigcup_{x \in A} F(x),$$
$$F^{-1}(B) := \{\, x \in X \mid F(x) \cap B \neq \emptyset \,\}.$$

A multivalued map $F\colon X \rightrightarrows Y$ is called *representable* over grids $\mathcal{G}$, $\mathcal{H}$ in X, Y respectively if it satisfies the following conditions:

(i) $\operatorname{card} \mathcal{G}(\operatorname{dom} F) < \infty$,

(ii) for every $x \in X$ the set $F(x)$ is representable,

(iii) if E is an elementary representable set then $F_{|E} = \mathrm{const}$.

Theorem 2.2 *(see [10]) If $A \in \mathrm{R}(\mathcal{G})$, $B \in \mathrm{R}(\mathcal{H})$ and F is a representable multivalued map, then*

$$\operatorname{dom} F,\ F^{-1}(B) \in \mathrm{R}(\mathcal{G}),$$
$$F(A) \in \mathrm{R}(\mathcal{H}).$$

Let $N \subset X$ be a compact representable set. Define

$$F_N\colon X \ni x \to N \cap F(x) \subset X.$$

Proposition 2.3 *F_N is representable.*

We say that a multivalued map $F\colon X \rightrightarrows Y$ is *upper semicontinuous* if for every $x \in X$ the set $F(x)$ is compact and for every neighborhood U of $F(x)$ there exists a neighborhood V of x such that $F(V) \subset U$.

If a sequence of multivalued maps $\{F_n\}$ is given then we say that this sequence converges to a multivalued map f, which we denote by $F_n \to f$, if the graphs of F_n converge to the graph of f as subsets of $X \times Y$ with respect to the Hausdorff metric.

A multivalued map f is called *singe-valued* if $\operatorname{card} f(x) \leq 1$ for every $x \in X$ and may be identified with a map $X \multimap Y$ defined on a subset of X.

A single-valued map $f\colon X \multimap Y$ is called a *selector* of a multivalued map $F\colon X \rightrightarrows Y$ if $f(x) \in F(x)$ for every $x \in \operatorname{dom} f$ (in particular, $\operatorname{dom} f \subset \operatorname{dom} F$).

Assume X, Y are two locally compact metric spaces with given grids $\mathcal{G}$, $\mathcal{H}$. Let $f\colon X \multimap Y$ be a continuous map defined on a subset of X. We say that $F\colon X \rightrightarrows Y$ is a *representation* of f if F is representable and f is a selector of F.

Theorem 2.4 *(see [9]) Assume $\mathcal{G}_n$, $\mathcal{H}_n$ are sequences of grids in X, Y respectively, such that $\operatorname{diam} \mathcal{G}_n \to 0$ and $\operatorname{diam} \mathcal{H}_n \to 0$. Let $f\colon X \multimap Y$ be a Lipschitz function such that $\operatorname{dom} f$ is relatively compact. Then there exist sequences of multivalued maps $F_n, G_n\colon X \rightrightarrows Y$ such that*

- *(i) F_n, G_n are representations of f,*
- *(ii) F_n is lower semicontinuous and G_n is upper semicontinuous,*
- *(iii) $F_n \to f$, $G_n \to f$.*

3 The Conley index

Let $f\colon \mathbb{R}^d \to \mathbb{R}^d$ be a homeomorphism.

If $N \subset \mathbb{R}^d$ then the set

$$\operatorname{Inv} N := \operatorname{Inv}(N, f) := \{x \in N \mid \forall n \in \mathbb{Z}\ f^n(x) \in N\}$$

is called the *invariant part* of N.

A compact set $N \subset \mathbb{R}^d$ is called an *isolating neighborhood* if

$$\operatorname{Inv} N \subset \operatorname{int} N.$$

A set $S \subset \mathbb{R}^d$ is called an *isolated invariant set* if there exists an isolating neighborhood N such that $S = \operatorname{Inv} N$.

A pair $P = (P_1, P_2)$ of compact subsets of an isolating neighborhood N is called an *index pair* if $P_2 \subset P_1$ and

(i) $x \in P_i,\ f(x) \in N \ \Rightarrow\ f(x) \in P_i, \quad i = 1, 2,$

(ii) $x \in P_1,\ f(x) \notin N \ \Rightarrow\ x \in P_2$,

(iii) $\operatorname{Inv} N \subset \operatorname{int}(P_1 \backslash P_2)$.

Let H^* denote the Alexander-Spanier cohomology functor. Let i_P be the inclusion $(P_1, P_2) \to (P_1 \cup f(P_2), P_2 \cup f(P_2))$. Since f maps (P_1, P_2) to $(P_1 \cup f(P_2), P_2 \cup f(P_2))$ and i_P is an excision for the Alexander-Spanier cohomology, we can define the *index map* according to the formula

$$I_P := H^*(f_P) \circ H^*(i_P)^{-1} \colon H^*(P_1, P_2) \to H^*(P_1, P_2).$$

Define the *generalized kernel* of this map as

$$\operatorname{gker}(I_P) := \bigcup_{n \in \mathbb{N}} \ker I_P^n.$$

The Conley index is then defined as

$$CH^*(S, f) := \left(H^*(P_1, P_2) / \operatorname{gker}(I_P), [I_P] \right),$$

where $[I_P]$ stands for the automorphism induced by I_P on the quotient space $H^*(P_1, P_2) / \operatorname{gker}(I_P)$.

Consider now the multivalued case. Let $F \colon \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ be an upper semicontinuous multivalued map and let $N \subset \mathbb{R}^d$ be a compact set. The invariant part of N is the set

$$\operatorname{Inv}(N, F) := \{ x \in N \mid \exists \sigma \colon \mathbb{Z} \to N \text{ such that } \sigma(0) = x \text{ and } \sigma(n+1) \in F(\sigma(n)) \}.$$

The set N is called an *isolating neighborhood* if

$$\operatorname{Inv} N \cup F(\operatorname{Inv} N) \subset \operatorname{int} N.$$

A pair $P = (P_1, P_2)$ of compact sets is an *index pair* in an isolating neighborhood N if $P_2 \subset P_1 \subset N$ and

(i) $F(P_i) \cap N \subset P_i,\ i = 1, 2$,

(ii) $F(P_1 \backslash P_2) \subset N$,

(iii) $\operatorname{Inv} N \subset \operatorname{int}(P_1 \backslash P_2)$.

The index map in this case is defined according to the formula

$$I_P := H^*(F_P) \circ H^*(i_P)^{-1} \colon H^*(P_1, P_2) \to H^*(P_1, P_2)$$

and the Conley index is defined as

$$CH^*(S, f) := (H^*(P_1, P_2)/\operatorname{gker}(I_P), [I_P]) .$$

Assume $\mathcal{A}$ is a collection of multivalued maps. We recall that property φ of maps in $\mathcal{A}$ is *inheritable* if for every $F \in \mathcal{A}$ and every selector f of F

$$\varphi(F) \;\Rightarrow\; \varphi(f).$$

We say that φ is *strongly inheritable* if φ is inheritable and for any single-valued map $f \in \mathcal{A}$ such that $\varphi(f)$ and for any sequence $\{F_n\} \subset \mathcal{A}$ satisfying $F_n \to f$ we have $\varphi(F_n)$ for n sufficiently large. Finally, if $\alpha(F)$ is a term then we say that α is *inheritable* (*strongly inheritable*) if for any x the property $\alpha(F) = x$ is inheritable (strongly inheritable).

Theorem 3.1 *(see [9]) Isolating neighborhood, index pair and Conley index are strongly inheritable terms.*

4 Existence of periodic orbits

Let $f\colon \mathbb{R}^d \to \mathbb{R}^d$ be a vector field on $\mathbb{R}^d$ of class C^1. Let $\varphi\colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ be the flow on $\mathbb{R}^d$ generated by the differential equation

(1) $$x' = f(x).$$

A compact subset Ξ of a $(d-1)$–dimensional hyperplane Π is called a *local section* for φ if the vector field f is transverse to Π on Ξ. Such a set Ξ is called a *Poincaré section* for φ in an isolating neighbourhood N if $\Xi \cap N$ is closed and for every $x \in N$ there exists $t > 0$ such that $\varphi(x, t) \in \Xi$.

Given a $t \in \mathbb{R}$, define the *time-t map* by

$$\varphi_t\colon \mathbb{R}^d \ni x \mapsto \varphi(x, t) \in \mathbb{R}^d.$$

Fix $t > 0$ and $\eta > 0$. Let N be a compact set representable with respect to the grid $\mathcal{G}_\eta$. Let F be a representation of φ_t on N. Assume

(2) $$F(N) \subset \operatorname{int} N.$$

In practice, we construct F and find an appropriate set N with the use of a method for rigorous integration of differential equations for some $t > 0$. We can expect that such a construction may be carried out in the case in which numerical simulations indicate the existence of an attracting periodic orbit. However, we must make sure that the chosen grid size η is small enough and the precision of the rigorous integration is high enough.

As a consequence of (2), the set N is an isolating neighborhood for F and the pair $P = (N, \emptyset)$ is an index pair for F in N.

Let S^1 denote the circle. Assume

(3) $$H^*(N) = H^*(S^1) \text{ and } I_P \text{ is an isomorphism.}$$

Then the Conley index of P is the index of an attracting periodic orbit. Due to the inheritability property, so is the Conely index of any selector of F, in particular of φ_t. Moreover, as a consequence of properties proved in [8], the Conley index of N with respect to the flow φ is also the index of an attracting periodic orbit. If so, it only remains to verify that

(4) $$N \text{ admits a Poincaré section,}$$

to have checked all the assumptions of the following theorem, proved in a more general setting in [4]:

Theorem 4.1 *Assume N is an isolating neighborhood for the flow φ which admits a Poincaré section Ξ. If N has the cohomological Conley index of a hyperbolic periodic orbit then* $\mathrm{Inv}(N, \varphi)$ *contains a periodic orbit.*

To sum up, our method is based on the following result:

Corollary 4.2 *Assume that for the flow generated by the differential equation (1) there exist t, η, F and N as described above, for which (2), (3) and (4) are satisfied. Then the differential equation (1) has a periodic solution.*

In practice, the verification of the assumptions of Corollary 4.2 involves a series of extensive, time-consuming computations. The algorithms which may be used for these computations are proposed in [12].

As a byproduct we obtain rigorous information concerning the location of the periodic orbit: it is contained in the interior of the isolating neighborhood N constructed in course of the computer assisted proof. Unfortunately, we do not prove anything about the period of this orbit. In particular, it is not ruled out that this orbit may make several turns along the neighborhood until it gets closed.

As an example consider the Rössler equations

$$
(5) \qquad \begin{cases} \dot{x} = -(y+z), \\ \dot{y} = x + by, \\ \dot{z} = b + z(x-a). \end{cases}
$$

For $a = 5.7$ and $b = 0.2$ the existence of chaos in (5) was proved in [16]. The chaotic attractor observed there seems to emerge via a series of period-doubling bifurcations of stable periodic orbits as the parameter a is increased. The first orbit in this series was observed in numerical simulations for $a = 2.2$ in [3], but the existence of a periodic orbit close to the observed one was proved only recently [13] with the use of the method described in this section. This method also allows to prove the existence of the second orbit, numerically best seen for $a = 3.1$ [14]. Summarizing, we can prove the following theorem.

Theorem 4.3 *Let $b = 0.2$. For $a = 2.2$ as well as for $a = 3.1$ the Rössler equations (5) admit a periodic orbit.*

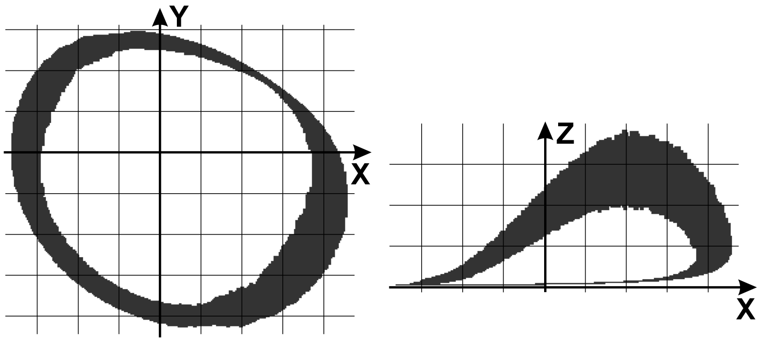

Figure 1: The isolating neighborhood constructed for the Rössler equations for the parameter value $a = 2.2$.

In Figure 1, projections to the XY and XZ planes of the neighborhood constructed for the Rössler equations (5) for the parameter value $a = 2.2$ are illustrated. The grid size used was $\eta = 1/32$. The time-step used $t = 3$ was approximately a half of the period of the periodic trajectory. The thin lines in the picture indicate integer coordinates. Note that a much tighter neighborhood may be obtained if a finer grid is taken, but then the computations are more costly in terms of computer time and memory used.

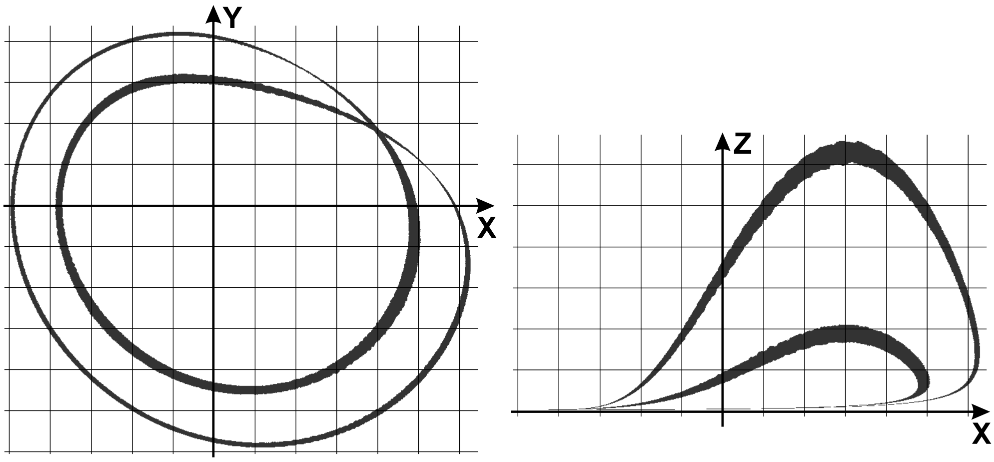

Figure 2: The neighborhood constructed for the Rössler equations with $a = 3.1$.

Figure 2 shows projections to the XY and XZ planes of the neighborhood computed for the periodic trajectory which numerically is observed to appear after the first period-doubling bifurcation in the Rössler equations. This neighborhood was created with the grid size $\eta = 1/256$ and the time-step $t = 2$.

All the rigorous computations needed to complete the proof of the existence of the periodic orbit took about 2 hours ($a = 2.2$) and 3 days ($a = 3.1$) on an IBM compatible PC running a 450 MHz processor.

As our second example consider the Lorenz equations

$$(6) \qquad \begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = Rx - y - xz, \\ \dot{z} = xy - bz. \end{cases}$$

For $R = 28$, $\sigma = 10$ and $b = 8/3$ the existence of chaos in these equations was proved in [2, 7]. However, when the parameter R is increased to $R = 260$ or to $R = 350$, attracting periodic orbits are observed in numerical simulations [15]: a symmetric one in the latter case and two mutually symmetric in the former case. These symmetries are due to the symmetry in the equations:

$$s : (x, y, z) \mapsto (-x, -y, z).$$

Our method allows to prove that there exist periodic orbits close to the location of the numerically observed ones. The details are presented in [14].

Theorem 4.4 *Fix* $\sigma = 10$ *and* $b = 8/3$*. For* $R = 260$ *the Lorenz equations (6) admit two mutually symmetric periodic orbits, and for* $R = 350$ *the Lorenz equations (6) admit a periodic orbit.*

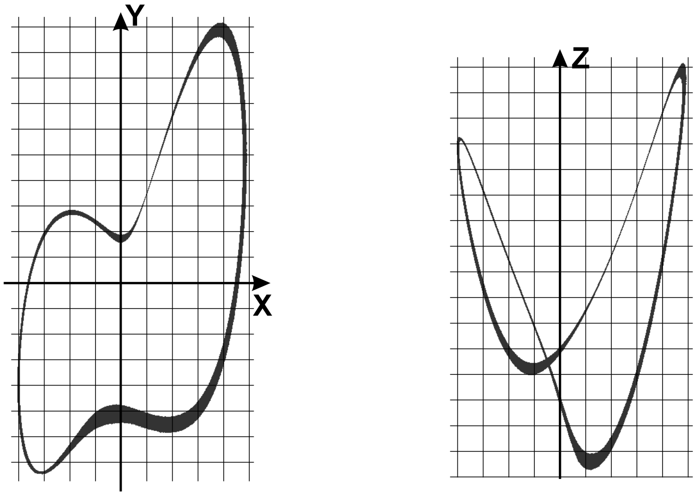

Figure 3: The isolating neighborhood constructed for the Lorenz equations for the parameter value $R = 260$.

In Figure 3, projections to the XY and XZ planes of a neighborhood of one of the two mutually symmetric periodic trajectories for $R = 260$ are plotted. The grid marked in the picture is drawn every 10 units. The Z coordinate of the bottom of the right-hand picture is 180. The grid size used in rigorous computations was $\eta = 1/16$. The time step was taken to be $t = 1/16$ which is about 1/7 of the approximate period of the trajectory observed in numerical simulations.

In Figure 4, projections to the XY and XZ planes of the neighborhood found for $R = 350$ are visualized. Again, the grid marked in the picture is drawn every 10 units. The Z coordinate of the bottom of the right-hand picture is 280. The grid size used in rigorous computations was $\eta = 1/8$. The time step was chosen as $t = 1/16$, which is about 1/6 of the approximate period of the observed trajectory.

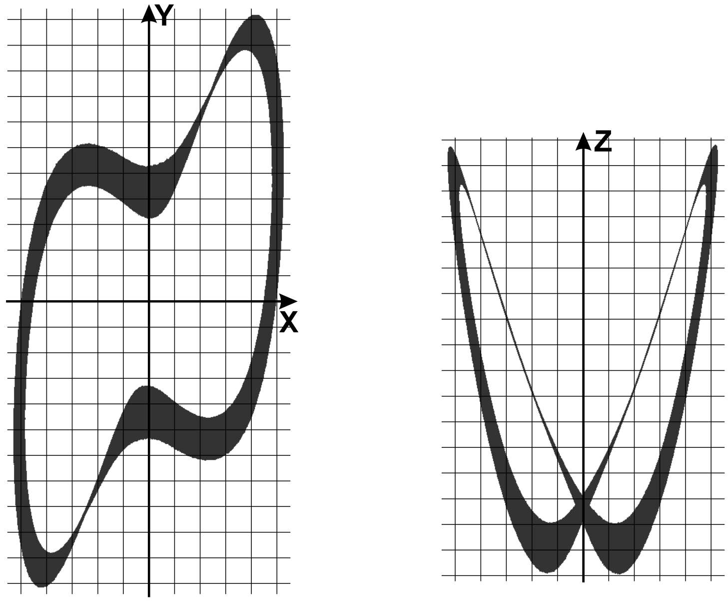

Figure 4: The neighborhood constructed for the Lorenz equations with $R = 350$.

The time of numerical computations on a IBM compatible PC running a 450 MHz processor amounted to about 4 days for the first orbit and 5 days for the other.

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